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VIBRATION AND STABILITY OF A NON-PRISMATIC COLUMN COMPRESSED BY NON-CONSERVATIVE FORCES IN NON-LINEAR CREEP CONDITIONS

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The paper deals with the problem of vibrations and stability of a non-prismatic column compressed by the follower force. The material of the column is characterized by Rabotnov's strain hardening non-linear creep law. It is assumed that the stress and strain in the basic state (e.g., pure compression) are subject to slight variation due to small vibrations. Thus, it is possible to linearize the creep law with respect to these variations so that the linear equations of motion can be obtained. They allow determination of the relationship between the real and imaginary parts of complex frequency and the compressive force (characteristic curves). The behaviour of characteristic curves for several types of non-prismatic columns have been examined and presented in numerous figures. Additionally, some parametrical optimization procedures have been performed.

1. INTRODUCTION

There exists a comprehensive literature devoted to the stability and vibrations of the linearly elastic structural systems, mainly columns [1]. Considerably fewer papers take into account the rheological properties of material. In particular, some of them consider a very interesting phenomenon of destabilization of non-conservative systems. Such phenomenon is due to the internal damping of material which has been characterized by the linear rheological model of a viscoelastic material of the Kelvin–Voigt type. Some references dealing with the above-mentioned problems have been presented by Gajewski [2]. Many other problems of analysis and synthesis of columns compressed by follower forces with respect to its stability have been discussed by Bogacz and Janiszewski [3] and Langthjem and Sugiyama [4–6].

Despite extensive literature devoted to the problems of stability of structural elements subjected to non-conservative forces (in particular the follower forces), Koiter [7] strongly criticized papers of that type. However, the phenomenon of the loss of stability of a column compressed by a follower force was experimentally verified for the first time by Yagn and Parshin [8] in 1967. New experiments have been performed by Sugiyama and co-workers [9, 10]. Recently, Langthjem *et al.* [11] during the Fourth EUROMECH Solid Mechanics Conference, June 26–30, 2000, Metz, France, in a very impressive manner, presented a video recording of their interesting experiments concerning columns subjected to rocket thrust and a pipeline conveying fluid. All experiments confirm the theoretical predictions that have been made so far. The experiments presented in references [9, 10] were carried out for viscoelastic columns with slight external and internal damping. As it is known, in such cases the critical force depends on the external to internal damping coefficients ratio and an external damping eliminates the destabilization effect (cf. references [12, 13]). It should be supposed that new experiments performed for structural elements made of creeping materials with great internal damping will confirm the results presented in this paper.

Stability of prismatic columns compressed by a non-conservative concentrated force in non-linear creep conditions was the object of research in the Ph.D. Thesis of Kowalski [14] (under the supervision of M. Życzkowski). Some elements of the work have been presented in reference [15]. The authors took into account Norton and Zhukov–Rabotnov–Churikov [16] non-linear models of material. The dependence of critical force causing the loss of static and kinetic stability on the so-called tangency coefficient has been considered.

A comprehensive review of papers (187 references) devoted to the optimal structural design under creep conditions is presented by Życzkowski [17]. Recently, an attempt at the optimization of a column compressed by a non-conservative force in non-linear creep conditions has been made by Gajewski [18].

The principal aim of the present work is to examine the influence of a non-prismatic form of the column on its vibrations and stability in non-linear creep conditions. Especially, the behaviour of characteristic curves will be considered. Some additional effects will also be taken into account, namely: (1) compressibility of the axis; (2) rotatory inertia; (3) shear deformations and (4) external damping. Similar problems for prismatic columns have been analyzed in reference [2].

2. NON-LINEAR CONSTITUTIVE EQUATIONS OF CREEP

In order to explain the main ideas of this paper, some statements from paper [2] will be repeated. It is assumed that the stress and strain components of the basic uniaxial stress state are interrelated by the following creep law, accounting for strain hardening:

$$\Phi(\bar{\sigma}, p, \dot{p}) = 0, \quad \text{where} \quad p = \varepsilon - \bar{\sigma}/\bar{E}, \tag{1}$$

the symbol p denotes the creep strain, ε the full strain, $\bar{\sigma}$ the stress, $\bar{E}$ the elastic modulus and Φ is a given material function, a bar over a symbol denotes dimensional quantities and a dot denotes differentiation with respect to time. Generally, it can be assumed that during vibrations of a system (or as a result of buckling), the stress and strain components in the basic state are subjected to small variations and creep law (1) can be linearized with respect to them. The behaviour of the vibrations determines the stability of the basic state (precritical) at a critical time $\bar{t}_*$. In the basic state, the relation between stress $\bar{\sigma}$ and strain ε is determined by equation (1), while the "tangent creep modulus" $\bar{E}_{tc}$ should be evaluated on the basis of Rabotnov–Shesterikov [19] theory from the equation

$$\partial\Phi/\partial\dot{p}|_0\, \delta\dot{p} + \partial\Phi/\partial p|_0\, \delta p + \partial\Phi/\partial\bar{\sigma}|_0\, \delta\bar{\sigma} = 0. \tag{2}$$

Assuming that the variations of stress and strain components are subjected to small linear vibrations of complex frequency $\bar{\Omega}$

$$\delta\varepsilon = \delta\varepsilon^a\, \mathrm{e}^{\bar{\Omega}\bar{t}}, \qquad \delta\bar{\sigma} = \delta\bar{\sigma}^a\, \mathrm{e}^{\bar{\Omega}\bar{t}}, \tag{3}$$

and by substituting them into equation (2) one obtains the tangent creep modulus

$$\bar{E}_{tc} = \frac{\delta\bar{\sigma}}{\delta\varepsilon} = \frac{\delta\bar{\sigma}^a}{\delta\varepsilon^a} = \frac{\bar{\Omega}\ \partial\Phi/\partial\dot{p}|_0 + \partial\Phi/\partial p|_0}{(\bar{\Omega}/\bar{E})\, \partial\Phi/\partial\dot{p}|_0 + (1/\bar{E})\, \partial\Phi/\partial p|_0 - \partial\Phi/\partial\bar{\sigma}_0|}\,. \tag{4}$$

In this paper, the commonly used Rabotnov's strain hardening creep law has been adopted:

$$\Phi = \dot{p} - \Gamma\,\bar{\sigma}^n/p^\mu = \dot{p} - \Gamma\bar{\sigma}^n p^{-\mu} \tag{5}$$

where μ, n, Γ denote material constants (dependent on temperature). Moreover, all results are obtained using material constants for copper at 200°C, $n = 32{\cdot}8$, $\mu = 9{\cdot}52$, $\bar{E} = 1{\cdot}22 \times 10^5$ MPa, $\Gamma = 2{\cdot}18 \times 10^{-113+n}\,(\mathrm{MPa})^{-n}\mathrm{h}^{-1}$ (see Zhukov *et al.* [16]). In the basic precritical state under assumptions of constant stress $\bar{\sigma} = \mathrm{const}(t)$ and initial condition $p(0) = 0$, one obtains from equations (1) and (5)

$$\varepsilon_0 = (\bar{\sigma}_0/\bar{E})\,\{1 + \bar{E}[(1+\mu)\Gamma\bar{t}_*]^{1/(1+\mu)}|\bar{\sigma}_0|^{(n-1-\mu)/(1+\mu)}\} \tag{6}$$

and the "secant modulus":

$$E_{sc} = \bar{E}_{sc}/\bar{E}_0 = \bar{E}/\bar{E}_0\,\{1 + \bar{E}[(1+\mu)\Gamma\bar{t}_*]^{1/(1+\mu)}|\bar{\sigma}_0|^{(n-1-\mu)/(1+\mu)}\}. \tag{7}$$

According to the Rabotnov–Shesterikov [19] theory, the "tangent modulus" for the non-linear creep law (5) can be written in the form of

$$E_{tc} = \frac{\bar{E}(1 + ((1+\mu)/\mu)\,\bar{t}_*\bar{\Omega})}{\bar{E}_0(1 + ((1+\mu)/\mu)\,\bar{t}_*\bar{\Omega} + (n(1+\mu)/\mu)\,\bar{t}_*\Gamma\bar{E}[(1+\mu)\Gamma\bar{t}_*]^{-\mu/(1+\mu)}|\bar{\sigma}_0|^{(n-1-\mu)/(1+\mu)})}. \tag{8}$$

It is a function of critical time $\bar{t}_*$ and of complex frequency of vibration $\bar{\Omega} = \bar{\delta} + \mathrm{i}\bar{\omega}$. $\bar{E}_0$ is a certain constant of the stress dimension.

3. BASIC EQUATIONS OF COLUMN VIBRATION AND KINETIC STABILITY

The general equations of the precritical and vibration states have been derived and presented in monograph [1] where the effects of (1) extensibility of the axis, (2) shear deformations, (3) rotary inertia and (4) non-linear properties of material have been introduced. The equations of small vibrations of complex frequency $\bar{\Omega}$ superposed on the momentless precritical state can be transformed into four complex or to eight real linear ordinary differential equations with the appropriate boundary conditions. The loss of stability at a critical force P_{cr} is determined by the condition: $\bar{\delta} = 0$.

In the case of a cantilever compressed by a follower force $\bar{P}$, as presented in Figure 1, the system of basic differential equations constitutes the non-self-adjoint boundary value problem

$$v' = (1+\varepsilon_{00})(1+hP)\varphi + (1+\varepsilon_{00})hQ, \qquad \varphi' = -M/E_{tc}I, \tag{9}$$

$$M' = (1+\varepsilon_{00})(1+hP)Q + P(1+\varepsilon_{00})(1+hP)\varphi - r\alpha\rho\Omega^2 I\varphi,$$

$$Q' = \rho\Omega^2 Av + \gamma\Omega Bv, \qquad v(0) = 0, \quad \varphi(0) = 0, \quad M(1) = 0, \quad (Q + \eta P\varphi)_1 = 0, \tag{10}$$

in which the dimensionless quantities have been introduced,
independent variables:

$$x = \bar{s}/\bar{l}, \qquad t = \bar{t}/\bar{t}_0;$$

state variables:

$$u = \bar{u}/\bar{l}, \quad v = \bar{v}/\bar{l}, \quad \varphi = \bar{\varphi}, \quad N = \bar{N}\bar{l}^2/\bar{E}_0\bar{I}_0, \quad Q = \bar{Q}\bar{l}^2/\bar{E}_0\bar{I}_0, \quad M = \bar{M}\bar{l}/\bar{E}_0\bar{I}_0;$$

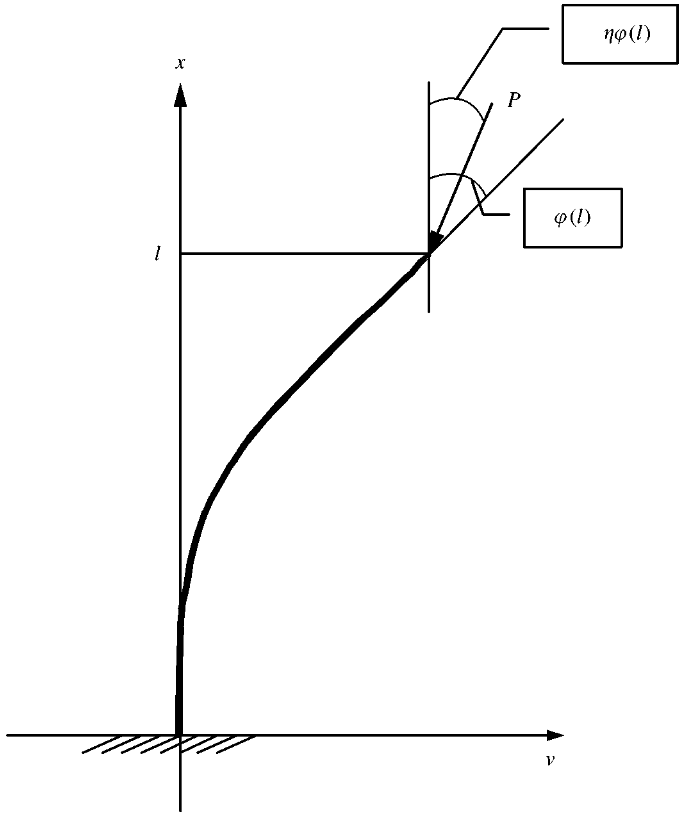

Figure 1. The cantilever column compressed by a non-conservative force.

possible design variables:

$$A(x) = \bar{A}/\bar{A}_0, \quad I(x) = \bar{I}/\bar{I}_0, \quad e(x) = \bar{E}/\bar{E}_0, \quad \rho(x) = \bar{\rho}/\bar{\rho}_0;$$

constant parameters:

$$\alpha = \bar{I}_0/\bar{A}_0\bar{l}^2, \qquad \bar{t}_0 = \sqrt{\bar{\rho}_0\bar{A}_0\bar{l}^4/\bar{E}_0\bar{I}_0}, \qquad t_* = \bar{t}_*/\bar{t}_0;$$

external force and frequency of vibration:

$$P = \bar{P}\bar{l}^2/\bar{E}_0\bar{I}_0, \quad \Omega = \bar{t}_0\bar{\Omega}, \quad \delta = \bar{t}_0\bar{\delta}, \quad \omega = \bar{t}_0\bar{\omega};$$

elongation of the column axis:

$$\varepsilon_{00} = -\alpha P/E_{sc}A, \qquad \varepsilon_{01} = \mathbf{1} + \varepsilon_{00}. \tag{11}$$

$\bar{A}_0$ and $\bar{I}_0$ denote the cross-sectional area and the moment of inertia of the cross-section of a prismatic column which has the same length and volume as the given non-prismatic bar, $\bar{E}_0$ and $\bar{\rho}_0$ are certain constants of stress and density dimensions, with $\bar{t}_0$ denoting a certain constant of time which may be treated as a unit of time. Additionally, quantities connected with the applied physical law have been introduced as

$$T = T_{00}\,(\bar{\tau}/\bar{\tau}_0)^{1/(1+\mu)}(\alpha/\alpha_0)^{(n-1-\mu)/(1+\mu)}, \qquad T_{00} = e[(1+\mu)\Gamma \bar{E}_0^n \bar{\tau}_0]^{1/(1+\mu)}\alpha_0^{(n-1-\mu)/(1+\mu)},$$

$$E_{sc} = e/(1 + T\,(P/A)^{(n-1-\mu)/(1+\mu)}), \qquad E_{tc} = \frac{e(1 + ((1+\mu)/\mu)\,t_*\Omega)}{1 + ((1+\mu)/\mu)\,t_*\Omega + (n/\mu)\,T\,(P/A)^{(n-1-\mu)/1+\mu)}},$$

$$\bar{\tau} = \bar{t}_0 t_* = \bar{t}_*, \bar{\tau}_0 = 3600\,(s) = 1(h), \quad \alpha_0 = 10^{-4}. \tag{12}$$

By considering the physical constants for copper, given above, and using e ≡ 1 one obtains $T_{00} = 0{\cdot}71408$.

In equations (9) and (10), the dimensionless frequency of vibrations $\Omega = \bar{t}_0\bar{\Omega}$ has been introduced, while η, γ, $B(x)$ denote the "tangency" coefficient, external viscous damping coefficient and dimensionless width of the column respectively. Parameter r characterizes the value of the cross-sectional rotatory inertia, while function $h(\Omega)$, characterizing shear effects, is defined by the equation

$$h(\Omega) = \alpha\varepsilon_{01}/(k_1 E_{tc} A - \alpha\varepsilon_{01} P), \tag{13}$$

where k_1 denotes the shear coefficient.

In order to use the transfer matrix numerical method, the system of complex equations (9) is rewritten in the form of a system of real equations. After introducing the notation for real and imaginary parts, namely,

$$v = v_1 + \mathrm{i}v_2, \qquad \varphi = \varphi_1 + \mathrm{i}\varphi_2, \qquad \Omega = \delta + \mathrm{i}\omega, \qquad h = h_1 + \mathrm{i}h_2,$$
$$1/E_{tc} = \tilde{E}_{tc} = \tilde{E}_1 + \mathrm{i}\tilde{E}_2, \tag{14}$$

equations (9) and (10) take the form

$$\begin{aligned}
v_1' &= \varepsilon_{01}(1 + h_1 P)\varphi_1 - \varepsilon_{01}h_2 P\varphi_2 + \varepsilon_{01}h_1 Q_1 - \varepsilon_{01}h_2 Q_2,\\
\varphi_1' &= -(1/I)\,\tilde{E}_1 M_1 + (1/I)\,\tilde{E}_2 M_2,\\
M_1' &= (P\varepsilon_{01}(1 + h_1 P) - r\alpha\rho(\delta^2 - \omega^2)I)\,\varphi_1 + \varepsilon_{01}(1 + h_1 P)Q_1\\
&\quad + (-P^2\varepsilon_{01}h_2 + 2r\alpha\rho\delta\omega I)\,\varphi_2 - \varepsilon_{01}h_2 P Q_2\\
Q_1' &= \rho(\delta^2 - \omega^2)Av_1 - 2\rho\delta\omega Av_2 + \gamma\delta Bv_1 - \gamma\omega Bv_2.\\
v_2' &= \varepsilon_{01}h_2 P\varphi_1 + \varepsilon_{01}h_2 Q_1 + \varepsilon_{01}(1 + h_1 P)\varphi_2 + \varepsilon_{01}\mathrm{h}_1 Q_2\\
\varphi_2' &= -(1/I)\,\tilde{E}_2 M_1 - (1/I)\,\tilde{E}_1 M_2,\\
M_2' &= (P^2\varepsilon_{01}h_2 - 2r\alpha\rho\delta\omega I)\,\varphi_1 + \varepsilon_{01}h_2 P Q_1\\
&\quad + (P\varepsilon_{01}(1 + h_1 P) - r\alpha\rho(\delta^2 - \omega^2)I)\,\varphi_2 + \varepsilon_{01}(1 + h_1 P)Q_2,\\
Q_2' &= 2\rho\delta\omega Av_1 + \rho(\delta^2 - \omega^2)Av_2 + \gamma\omega Bv_1 + \gamma\delta Bv_2,
\end{aligned} \tag{15}$$

$$v_1(0) = 0, \qquad v_2(0) = 0, \qquad \varphi_1(0) = 0, \qquad \varphi_2(0) = 0,$$

$$M_1(1) = 0, \qquad M_2(1) = 0, \qquad (Q_1 + \eta P\varphi_1)_1 = 0, \qquad (Q_2 + \eta P\varphi_2)_1 = 0, \tag{16}$$

where

$$a = ((1 + \mu)/\mu)\,\bar{\tau}/\bar{\tau}_0, \qquad b = 1 + (n/\mu)\,T\,(P/A)^{(n-1-\mu)/(1+\mu)},$$

$$\tilde{E}_1 = \frac{(1 + a\delta)(b + a\delta) + a^2\omega^2}{e[(1 + a\delta)^2 + a^2\omega^2]}, \qquad \tilde{E}_2 = \frac{(1 - b)a\omega}{e[(1 + a\delta)^2 + a^2\omega^2]},$$

$$h_1 = \frac{\alpha\varepsilon_{01}[k_1 A\tilde{E}_1 - \alpha\varepsilon_{01}P(\tilde{E}_1^2 + \tilde{E}_2^2)]}{[k_1 A - \alpha\varepsilon_{01}P\tilde{E}_1]^2 + [\alpha\varepsilon_{01}P\tilde{E}_2]^2}, \qquad h_2 = \frac{\alpha\varepsilon_{01}k_1 A\tilde{E}_2}{[k_1 A - \alpha\varepsilon_{01}P\tilde{E}_1]^2 + [\alpha\varepsilon_{01}P\tilde{E}_2]^2}. \tag{17}$$

The boundary value problems (15) and (16) have been integrated by means of the transfer matrix method which was widely used by Irie *et al.* [20]. It determines the so-called characteristic curves, i.e., the relations between load parameter P and the real (δ) and imaginary (ω) parts of complex frequency of vibration (Ω). In the case of non-conservative load, the vibrations of the column are stable if $\delta < 0$ and they lose stability by flutter if the real part of a frequency changes its sign.

4. PARAMETRICAL OPTIMIZATION, RESULTS AND DISCUSSION

A numerical analysis was performed for a cantilever column loaded by a follower force, as it is shown in Figure 1. The reference density $\bar{\rho}_0$ has been assumed to be equal to the column density $\bar{\rho}$ and as a result $\rho = 1$. Similarly, $e = 1$. All calculations were performed for slenderness parameter $\alpha = 10^{-5}$.

The dimensionless column cross-section area is normalized according to the constant volume condition

$$\int_0^1 A(x)\,\mathrm{d}x = \frac{\bar{V}}{\bar{A}_0\bar{l}} = 1, \tag{18}$$

where $\bar{V}$ is the volume of the column.

In this paper, the transfer matrix method is applied to columns with three types of cross-sectional area variation, (1) power function; (2) power function with added sine and (3) "optimal shape".

1. *Power function.* For the first case, the function $A(x)$ is calculated from the formula

$$A(x) = (s + 1)(1 - A_1 x^s)/(s + 1 - A_1), \tag{19}$$

where: $s \geqslant 0$, $A_1 \leqslant 1$. For example, for the column of linearly varying cross-sectional area, i.e., for $s = 1$, for the tangential force $\eta = 1$, for the case without external damping, rotatory inertia and shear effects ($\gamma = 0$, $k_1 \to \infty$ and $r = 0$), the characteristic curves corresponding only to the first frequency of vibration are presented in Figures 2 and 3. They are obtained for critical time parameters $\tau = 0{\cdot}01$ and $0{\cdot}1$, respectively, and for various values of shape parameter A_1. In Figure 2, one can see that for the lower value of critical time parameter the critical forces (corresponding to $\delta_1 = 0$) are very similar to one another, although the curves P versus ω_1 are rather different. For the greater value of critical time parameter $\tau = 0{\cdot}1$, the critical forces are strongly dependent on the shape parameter A_1. The maximal critical force $P_{cr} = 10{\cdot}89$ is obtained for $A_1 = -0{\cdot}4$ (see Figure 3).

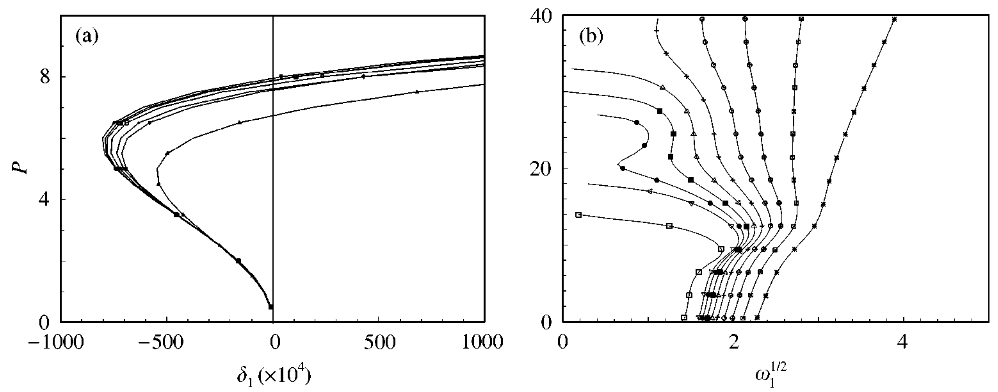

Figure 2. Characteristic curves of (a) P versus δ_1 and (b) P versus ω_1 for a column of linearly varying cross-section with parameters $\tau = 0{\cdot}01$, $s = 1$, $\eta = 1$, $\alpha = 10^{-5}$, $\gamma = 0$ and $k_1 \to \infty$. Key for column shapes (A_1): —□—, − 2·0; —▽—, − 1·0; —●—, − 0·8; —■—, − 0·6; —▲—, − 0·4; —+—, − 0·2; —○—, 0·0; —⊕—, 0·2; —■—, 0·4; —*—, 0·6.

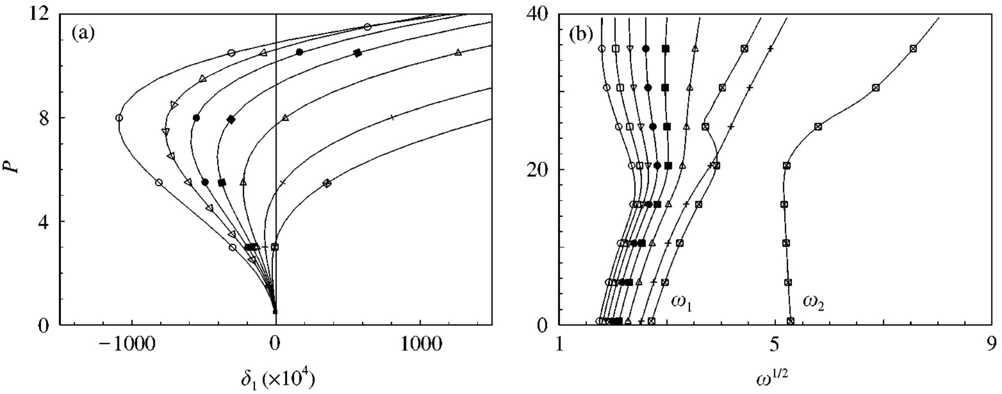

Figure 3. Characteristic curves of (a) P versus δ_1 and (b) P versus ω_1 for a column of linearly varying cross-section with parameters $\tau = 0{\cdot}1$, $s = 1$, $\eta = 1$, $\alpha = 10^{-5}$, $\gamma = 0$ and $k_1 \to \infty$. Key for column shapes (A_1): —⊠—, 0·9; —+—, 0·8; —△—, 0·6; —■—, 0·4; —●—, 0·2; —▽—, 0·0; —□—, − 0·2; —○—, − 0·4.

Further parametrical optimization by means of changing the parameter s in function (19) increases the value of critical force only to 11·16 for $s = 0{\cdot}4$ and $A_1 = -0{\cdot}4$.

2. *Power function with sine.* For the second type of shape variation, one confines the calculations to the linear function with added sine function, namely

$$A(x) = 2(1 - A_1 x)/(2 - A_1) + A_2 \sin(2\pi x), \tag{20}$$

where the constants A_1 and A_2 should be chosen so that $A(x) > 0$. The column shapes analyzed in the paper are presented in Figure 4.

The maximal critical force $P_{cr} = 13{\cdot}30$ has been obtained for $A_2 = -0{\cdot}55$. The corresponding characteristic curves related to the first complex frequency of vibration are depicted in Figure 5.

3. *Optimal shape.* The third case of the column cross-section variation was obtained in two steps. At first, following Ringertz [21], a linear interpolation of the design A^G and the

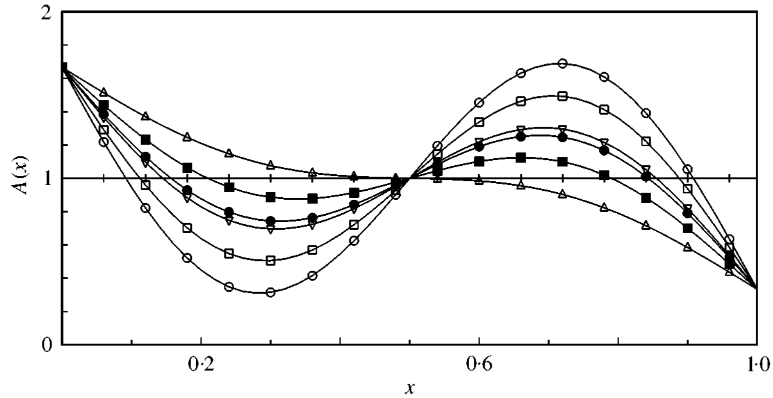

Figure 4. Shapes of the column cross-section corresponding to equation (20). Key for column shapes (A_2): —△—, − 0·2; —■—, − 0·4; —●—, − 0·55; —▽—, − 0·6; —□—, − 0·8; —○—, − 1·0.

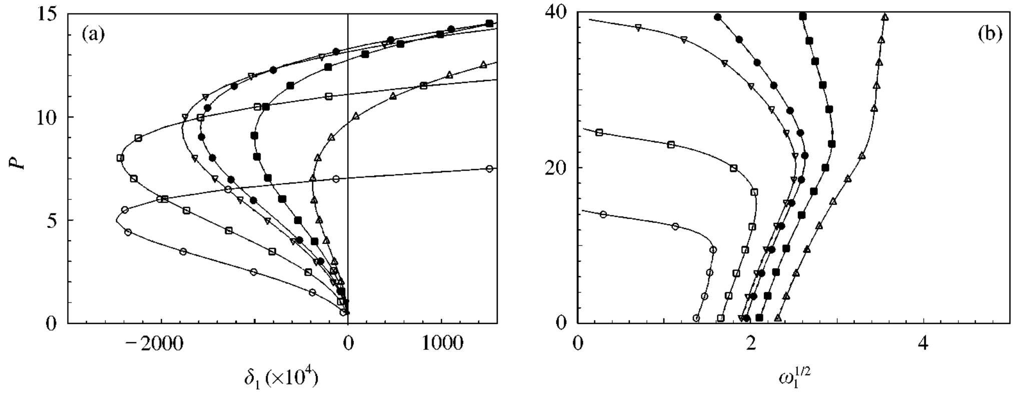

Figure 5. Characteristic curves of (a) P versus δ_1 and (b) P versus ω_1 for the column with shape variations listed in Figure 4. and with parameters $\tau = 0{\cdot}1$, $s = 1$, $A_1 = 0{\cdot}8$, $\eta = 1$, $\alpha = 10^{-5}$, $\gamma = 0$ and $k_1 \to \infty$.

initial uniform design A^0 was performed, such that

$$A^I(\xi) = (1 - \xi)\, A^0 + \xi A^G, \tag{21}$$

where ξ is a scalar parameter. The value $\xi = 0$ gives the uniform column and $\xi = 1$ gives the optimal column obtained by Gutkowski *et al.* [22] for an elastic material and for the tangential force ($\eta = 1$). In the present case of a non-linear material in creep conditions, the maximal critical force calculated for $\eta = 1$, $\tau = 0{\cdot}1$, $\alpha = 10^{-5}$, $\gamma = 0$, $k_1 \to \infty$, $r = 0$ has been obtained for $\xi = 0{\cdot}896$. In the second step, the design $A^I(0{\cdot}896)$ was approximated to the design A^A by the ninth degree polynomial. Finally, the so-called optimal shape was calculated from the formula

$$A = (1 - \xi_1) A^0 + \xi_1 A^A, \tag{22}$$

where $\xi_1 = 1{\cdot}003$. The result is presented in Figure 6.

The characteristic curves corresponding to four frequencies of vibration for the optimal shape are shown in Figure 7. Of course, the characteristic curves depend also on other

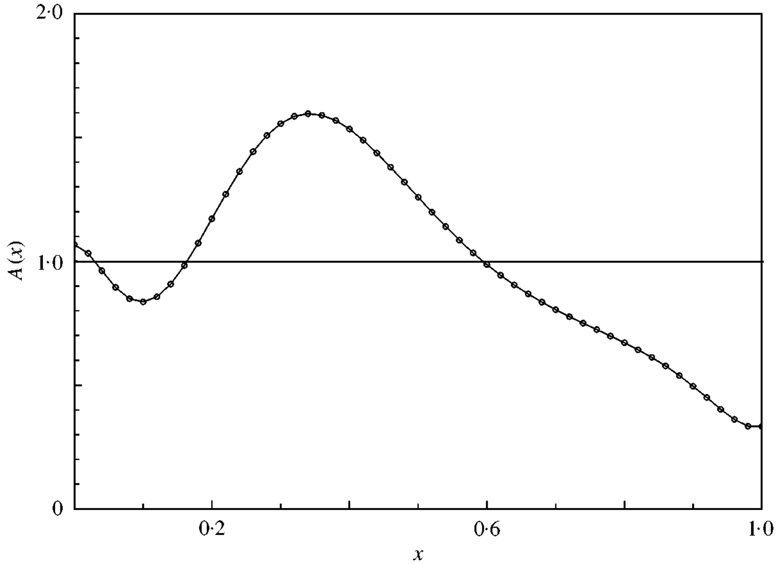

Figure 6. The "optimal shape".

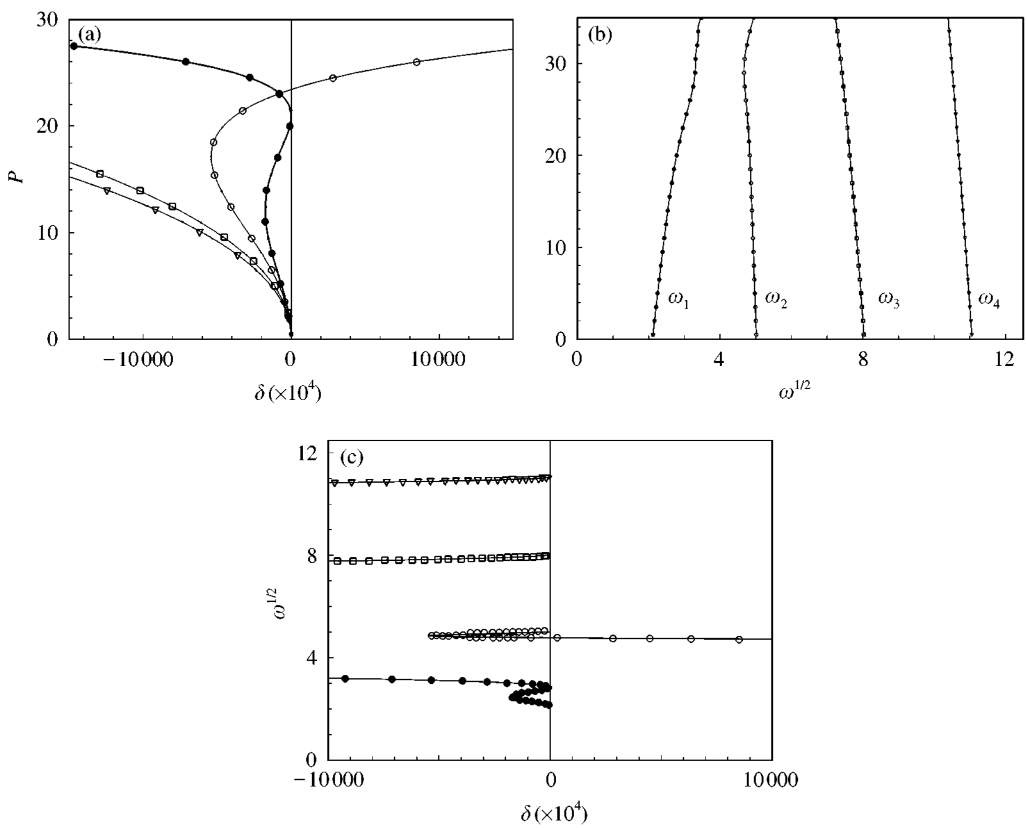

Figure 7. Characteristic curves of (a) P versus δ, (b) P versus ω and (c) ω versus δ for the four frequencies of vibration corresponding to the "optimal shape" and with parameters $\eta = 1$, $\tau = 0{\cdot}1$, $\alpha = 10^{-5}$, $\gamma = 0$ and $k_1 \to \infty$. Key for δ values: —●—, δ_1; —○—, δ_2; —□—, δ_4 and —▽—, δ_4. Key for ω values: —●—, ω_1; —○—, ω_2; —□—, ω_3 and —▽—, ω_4.

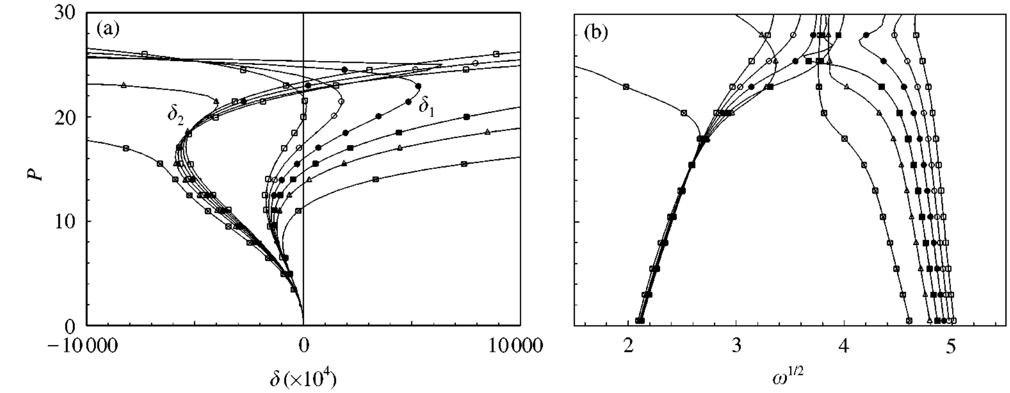

Figure 8. The influence of shear effect on the first and second vibrational frequencies for columns with "optimal" cross-section and having parameters $\eta = 1$, $\alpha = 10^{-5}$, $\tau = 0{\cdot}1$ and $\gamma = 0$. Key for shear values (k_1): —⊠—, 0·001; —△—, 0·002; —■—, 0·003; —●—, 0·005; —○—, 0·010 and —□—, 0·100.

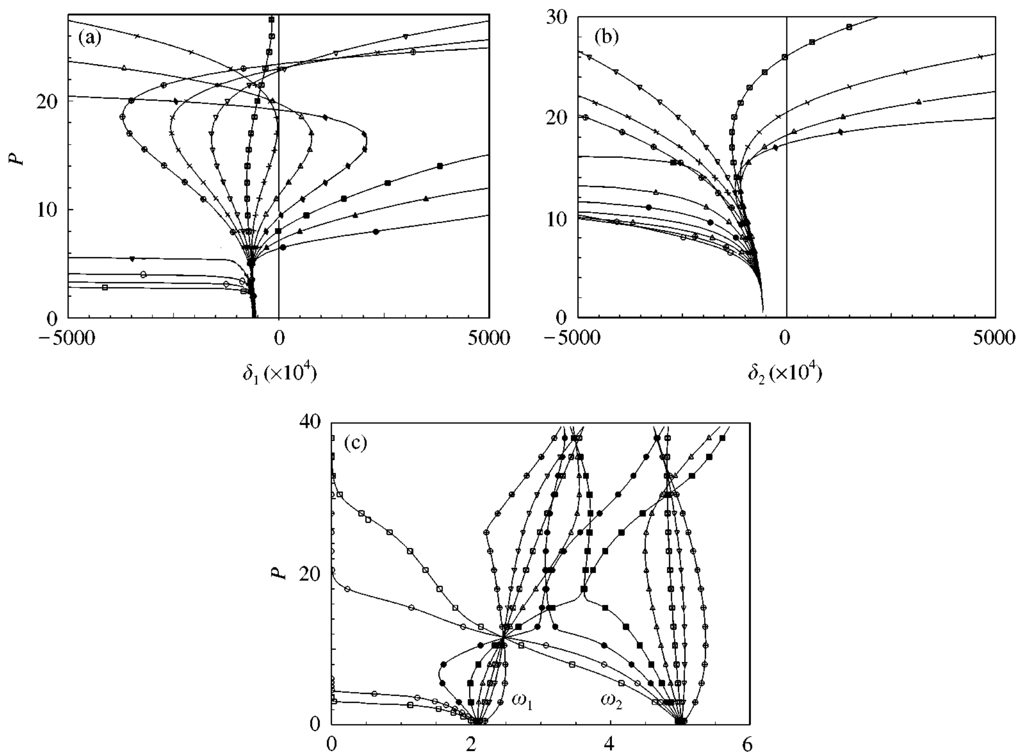

Figure 9. Characteristic curves for the optimally shaped column (having parameters $\alpha = 10^{-5}$, $\tau = 1{\cdot}0$, $\gamma = 0{\cdot}1$ and $k_1 = 0{\cdot}1$) versus the "tangency" coefficient, η. Key for "tangency" coefficient values (η): —□—, 0·0; —◇—, 0·1; —○—, 0·2; —▽—, 0·3; —●—, 0·4; —▲—, 0·5; —■—, 0·6; —◆—, 0·7; —△—, 0·8; —+—, 0·9; —⊠—, 1·0; —▽—, 1·2; —×—, 1·5; —⊕—, 2·0.

parameters of the problem, for example shear coefficient k_1, tangency coefficient η, external damping γ and slenderness parameter α. The influence of shear effects (coefficient k_1) on the first and second frequency of vibration are presented in Figure 8.

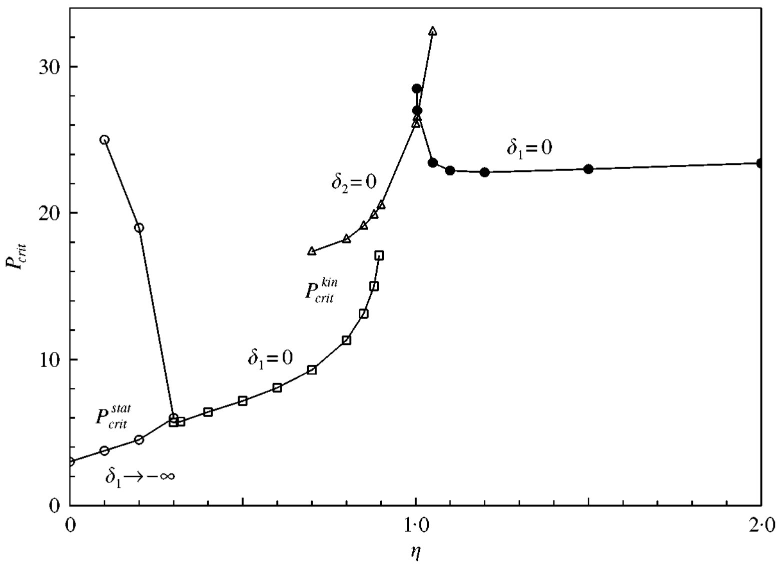

Figure 10. The critical forces versus "tangency" coefficient, for an optimally shaped column with parameters $\alpha = 10^5$, $\tau = 1{\cdot}0$, $\gamma = 0{\cdot}1$ and $k_1 = 0{\cdot}1$.

Finally, the behaviour of characteristic curves versus tangency coefficient η have been considered taking into account shear effects ($k_1 = 0{\cdot}1$) and external damping ($\gamma = 0{\cdot}1$). The results of the calculations are presented in Figure 9.

On the basis of the above data, one can construct the graph of critical force P_{cr} versus tangency coefficient η which is shown in Figure 10.

5. CONCLUSIONS

The behaviour of characteristic curves obtained for several types of cross-section variations of the column compressed by a non-conservative force have been presented. It was assumed that the material of the column is characterized by the non-linear creep law. On the basis of the calculations presented in numerous figures one can present the following conclusions:

(1) The general behaviour of characteristic curves obtained for non-prismatic columns differs from those presented in paper [2] for a prismatic column.

(2) The values of critical forces for the higher critical time parameter are to a higher degree dependent on the column shape (Figure 3).

(3) Although the non-linear creep law considered in this paper is essentially different from the linear viscoelastic models of material, the destabilizing effect presented here is close in character to the one obtained for the Kelvin–Voigt model (see e.g., reference [5]).

(4) As yet, the destabilizing effect of damping has not been verified experimentally (see e.g., [9–11, 13]). However, in the opinion of the author this effect should be taken into account in further considerations. As it is seen in Figures 7–9 the real parts δ_1 or δ_2 reach high positive values, which can lead to the loss of stability even in a short time (not only after infinite time).

(5) The "optimal shape" considered in the paper can be treated as a bimodal solution in the sense of equation $\delta_1 = \delta_2$ with $\omega_1 \neq \omega_2$ (see Figure 10).

(6) The maximal critical force obtained in the paper, $P_{cr} \approx 26$, is about 2·5 times greater than the critical force for a prismatic column (for $\eta = 1$), while for linearly elastic optimal shape the maximal critical force obtained so far, $P_{cr-el} = 188{\cdot}07$ (see reference [21]), is 9·4 times greater than for Beck's column.

(7) The graph presented in Figure 10 has a discontinuity at the point $\eta \approx 0{\cdot}9$, where the character of the loss of kinetic stability changes from the first to the second frequency of vibration. Such an effect has not been observed as yet.

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